1. **Basic notions of functions** A function is a map from a subset D of $\mathbb{R}$ to $\mathbb{R}$. The subset D is called the domain of f. The set $\{(x, f(x)) \colon x \in D\} \subset \mathbb{R}^2$ is called the graph of f. The set $\{f(x) \colon x \in D\}$ is called the range of f.

Exercise Write down domain, range of the following functions:

1a. $f(x) = \frac{1}{\sqrt{(|x|-2)(x^2+1)}}$;

Solution The domain is given by $\{x \colon \sqrt{(|x|-2)(x^2+1)} \neq 0\}$. To turn this into simple expression we have

$$\begin{aligned} & \{x \colon \sqrt{(|x|-2)(x^2+1)} \neq 0\} \\ = & \{x \colon \sqrt{(|x|-2)(x^2+1)} > 0\} \\ = & \{x \colon |x| - 2 > 0\} \\ = & (-\infty, -2) \cup (2, +\infty). \end{aligned}$$

The range of f is $(0, +\infty)$.

1b. $f(x) = \log_{10}(\cos x)$

Solution The domain is $\{x \colon \cos x > 0\}$ i.e. $\{x \colon -\frac{\pi}{2} + 2k\pi < x < \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\}$. Since $0 < \cos x \leqslant 1$, so $f(x) \leqslant 0$ i.e. the range of f is $(-\infty, 0]$.

A subset is called symmetric if both $x \in D$ and $-x \in D$. A function on such D is called an odd function if $f(x) + f(-x) = 0$. A function on such D is called even if $f(x) = f(-x)$. A typical odd function is $f(x) = x$ on $\mathbb{R}$. A typical even function is $f(x) = |x|$.

Exercise Is the following set symmetric?

1c. $D = \{x \colon (|x| - 1)x > 0\}$;

Solution We solve the inequality $(|x| - 1)x > 0$ explicitly:

$$\begin{aligned} & \{x \colon (|x|-1)x > 0\} \\ = & \{x \colon x > 0; |x| > 1\} \cup \{x \colon x < 0, |x| < 1\} \\ = & \{x \colon x > 1\} \cup \{x \colon -1 < x < 0\} \\ = & (1, +\infty) \cup (-1, 0). \end{aligned}$$

Obviously, D is not symmetric.

Or we can see that $2 \in D$, but $-2 \notin D$. This also gives D is not symmetric.

1d. $D = \{x: \cos x \geqslant 0\}$.

Solution Given $x \in D$, we also have $-x \in D$ since $\cos(-x) = \cos x \geqslant 0$. So D is symmetric.

Exercise Give the domain of the following function. Also is the following function even or odd?

1e. $f(x) = \frac{1}{\sqrt{(|x|-2)(x^2+1)}}$ on the set given by **1a**;

Solution $f(x) = f(-x)$ since $|x| = |-x|$ and $x^2 = (-x)^2$. So f is even.

1f. $f(x) = \cos x \sin x \sin 2x \sin 3x$;

Solution Since

$$\begin{aligned} f(-x) =& \cos(-x)\sin(-x)\sin(-2x)\sin(-3x) \\ =& \cos x\,(-\sin x)(-\sin 2x)(-3x) \\ =& -\cos x \sin x \sin 2x \sin 3x \\ =& -f(x). \end{aligned}$$

So f is odd.

1g. $f(x) = \frac{|\sin 5x|}{\sin 2x}$;

Solution

$$\begin{aligned} f(-x) =& \frac{|\sin(-5x)|}{\sin(-2x)} \\ =& \frac{|-\sin 5x|}{-\sin 2x} \\ =& \frac{|\sin 5x|}{-\sin 2x} = -f(x). \end{aligned}$$

So f is odd.

Exercise If f_1 and f_2 are odd, g_1 and g_2 are even, they are defined on the same symmetric domain. Is the following function even or odd?

1h. $f_1(x) f_2(x)$ **1i.** $f_1(x) g_1(x)$ **1j.** $g_1(x) g_2(x)$ **1k.** $f_1(f_2(x))$

Solution for 1h. $f_1(-x) f_2(-x) = [-f_1(x)][-f_2(x)] = f_1(x) f_2(x)$, so $f_1(x) f_2(x)$ is even.

Solution for 1i. $f_1(-x)g_1(-x) = -f_1(x)g_1(x)$, so $f_1(x)g_1(x)$ is odd.

Solution for 1j. $g_1(-x)g_2(-x) = g_1(x)g_2(x)$, so $g_1(x)g_2(x)$ is even.

Solution for 1k $f_1(f_2(-x)) = f_1(-f_2(x)) = -f_1(f_2(x))$, so $f_1(f_2(x))$ is odd.

2. **Limit of functions** A real number l is a limit of $f(x)$ at $x = a$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - l| < \varepsilon$$

and write

$$\lim_{x \to a} f(x) = l.$$

Sequential criterion for limits of functions $\lim_{x \to a} f(x) = l$ if and only if for any sequence x_n with $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} f(x_n) = l$.

We have the following **properties** of limits: Both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists, and let c be any number. Then

→ $\lim_{x \to a} [f(x) + cg(x)] = \lim_{x \to a} f(x) + c \lim_{x \to a} g(x)$;

→ $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$;

→ $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$.

Exercises It is known that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, do the following:

2a. Let $f(x) = \frac{|\sin 3x|}{x}$, let $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n}$, calculate

$$\lim_{n \to \infty} f(a_n) \quad \text{and} \quad \lim_{n \to \infty} f(b_n).$$

Then does the limit $\lim_{x \to 0} f(x)$ exist?

Solution Since $0 = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$, according to the sequential criterion and using $\lim_{x \to 0} \frac{\sin x}{x} = 1$, we have

$$\lim_{n \to \infty} \frac{\sin 3a_n}{3a_n} = \lim \frac{\sin 3b_n}{3b_n} = 1.$$

Since for n large, $0<\frac{1}{n}<\pi/3$, so

$$\begin{aligned}\lim_{n\to\infty} f(a_n) = &\lim_{n\to\infty}\frac{|\sin 3a_n|}{a_n}\\ =&\lim_{n\to\infty}\frac{\sin 3a_n}{a_n}\\ =&\lim_{n\to\infty}\frac{\sin 3a_n}{3a_n}\cdot 3\\ =&3>0.\end{aligned}$$

Similary, we have

$$\begin{aligned}\lim_{n\to\infty} f(b_n) = &\lim_{n\to\infty}\frac{|\sin 3b_n|}{a_n}\\ =&\lim_{n\to\infty}\frac{-\sin 3b_n}{b_n}\\ =&\lim_{n\to\infty}\frac{-\sin 3b_n}{3b_n}\cdot 3\\ =&3>0.\end{aligned}$$

$$\lim_{n\to\infty} f(b_n) = -3 \neq \lim_{n\to\infty} f(a_n).$$

So the limit $\lim_{x\to 0} f(x)$ does not exist.

2b. Use also previous trignometric formulas, calculate

$$\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}.$$

Solution

$$\begin{aligned}&\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}\\ =&\lim_{x\to 0}\frac{\frac{\sin x}{\cos x} - \sin^3 x}{\sin^3 x}\\ =&\lim_{x\to 0}\frac{1-\cos x}{\cos x \sin^2 x}\\ =&\lim_{x\to 0}\frac{\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2}) - [\cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})]}{\sin^2 x\cos x}\\ =&\lim_{x\to 0}\frac{2\sin^2(\frac{x}{2})}{(\frac{x}{2})^2}\cdot\frac{1}{\cos x}\cdot\frac{(\frac{x}{2})^2}{x^2}\cdot\frac{x^2}{\sin^2 x}\\ =&2\cdot 1\cdot\frac{1}{4}\cdot 1 = \frac{1}{2}.\end{aligned}$$

2c. A function f is called **continuous** at a if $\lim_{x\to a} f(x) = f(a)$. Is the function

$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

continous?

<u>Solution</u> Take any sequence a_n with $\lim_{n\to 0} a_n = 0$ where $a_n \neq 0$ for all n. Then $\lim_{n\to 0} |a_n| = 0$. Since $-|a_n| \leqslant a_n \sin a_n \leqslant |a_n|$, by squeez theorem,

$$\begin{aligned}\lim_{n\to\infty} f(a_n) &= \lim_{n\to\infty} a_n \sin\frac{1}{a_n} \\ &= 0 = f(0).\end{aligned}$$

By sequential criterion, this says that $\lim_{x\to 0} f(x) = 0 = f(0)$, so f is continuous at 0. f is continuous obviously elsewhere, so f is continuous.

<u>Another solution to 2c</u>

Since $-|x| \leqslant x\sin\frac{1}{x} \leqslant |x|$ and $\lim_{x\to 0} |x| = 0$, by sandwich theorem

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x} = 0 = f(0).$$

So f is continuous at 0. f is obviously continuous elsewhere.